JOURNAL OF APPROXIMATION THEORY 69, 318–325 (1992)

Counter-examples to Markov and Bernstein Inequalities

P. GOETGHELUCK

Département de Mathématiques, Université de Paris-Sud, Bât. 425, 91405 Orsay Cedex, France

AND

W. PLEŚNIAK

Institute of Mathematics, Jagiellonian University, Reymonta 4, 30-059 Kraków, Poland

Communicated by Vilmos Totik

Received November 26, 1990; revised April 3, 1991

We show that for any function $\varphi: \mathbb{N} \to \mathbb{R}^+$ one can find a Cantor set C and a trigonometric polynomial T of order $d(T)$ such that the generalized Bernstein inequality $\|T'\|_C \leqslant \varphi(d(T)) \|T\|_C$ does not hold. Furthermore, if $\varphi(n) \leqslant Mn^\rho$ (for some M and $\rho > 1$), the set C can be chosen to be regular with respect to the Green's function of $\mathbb{C} \setminus C$ with pole at ∞. Analogous results are established for algebraic polynomials and Markov's inequality. © 1992 Academic Press, Inc.

1. INTRODUCTION

It is well known that Markov's inequality plays an important role in the constructive theory of functions. Recently it has appeared that it is also closely related to the existence of a continuous linear operator extending C^∞ functions from compact subsets of $\mathbb{R}^N$ (see [3, 5]). Some new families of sets in $\mathbb{R}^N$ on which Markov's inequality holds have been found in [2] (case of L^∞-norm) and in [1] (case of L^p-norm). In [6], a Cantor type subset of $\mathbb{R}$ was constructed on which Markov's inequality fails to hold. In the one-dimensional case, Markov's inequality is closely related to Bernstein's inequality for trigonometric polynomials. This raises a similar question about counter-examples to the latter inequality. In this paper, we construct such counter-examples in a more general setting.

For more references on Markov's inequality see [7] (in the one-dimensional case) and the bibliography of [2] (in the case of several variables).

0021-9045/92 $5.00

For any real function f defined on $A \subset \mathbb{R}$ we put $\|f\|_A = \mathrm{Sup}_{x \in A} |f(x)|$. Let $A \subset [-\pi, \pi]$, $B \subset [-1, 1]$, and $\varphi: \mathbb{N} \to \mathbb{R}^+$ be given.

DEFINITION. We say that *the Bernstein inequaity holds on A with coefficient φ* if for any trigonometric polynomial T of order at most n, we have

$$\|T'\|_A \leqslant \varphi(n) \|T\|_A,$$

and, analogously, that *the Markov inequality holds on B with coefficient φ* if for any algebraic polynomial P of degree at most n we have

$$\|P'\|_B \leqslant \varphi(n) \|P\|_B.$$

The aim of this paper is to prove the following

THEOREM. *For any function $\varphi: \mathbb{N} \to \mathbb{R}^+$, there exists a Cantor set $C \subset [-\pi/2, \pi/2]$ ($C \subset [-1, 1]$, resp.) such that the Bernstein inequality (Markov's inequality, resp.) does not hold on C with coefficient φ. Furthermore, if $\varphi(n) \leqslant Mn^\rho$ (for some M and $\rho > 1$), the set C can be chosen to be regular with respect to Green's function of $\mathbb{C} \setminus C$ with pole at ∞.*

The paper is organized as follows: in Section 2, given an integer k and a function $\varphi: \mathbb{N} \to \mathbb{R}^+$, a sequence $u = (u_n)$ depending on φ and k and a Cantor set $C(k, \varphi)$ are constructed; Section 3 gives a sufficient condition in order that $C(k, \varphi)$ be regular; in Section 4, we prove the theorem using $C(k, \varphi)$ as a counter-example to the Bernstein inequality and (by a change of variable) to Markov's inequality.

2. CONSTRUCTING THE SEQUENCE (u_n) AND THE FAMILY $C(k, \varphi)$ OF CANTOR SETS

Let k be an integer satisfying $k \geqslant 5$ and $k \equiv 1 \pmod 4$. Given a function $\varphi: \mathbb{N} \to \mathbb{R}^+$, a sequence $u = (u_n)$ of positive integers is constructed inductively in the following way:

(i) $u_0 = 1$,

(ii) u_{n+1} is chosen so that

$$u_n \text{ divides } u_{n+1} \quad \text{and} \quad u_{n+1} \equiv 1 \pmod 4 \tag{2.1}$$

$$\mathrm{Max}\{5u_n, n\varphi(k^{n+1}u_n)/k^{n+1}\} \leqslant u_{n+1} \leqslant \mathrm{Max}\{5u_n, n^2\varphi(k^{n+1}u_n)\}. \tag{2.2}$$

We put $C_0=[-\pi/2, \pi/2]$. In order to construct the Cantor set $C(k, \varphi)$ we define by induction a sequence of compact subsets C_n of C_0 by

$$C_{n+1}=\{\theta\in C_n \mid 0\leqslant \cos(k^{n+1}u_n\theta)\leqslant \sin(\pi u_n/u_{n+1})\}.$$

The set C_n $(n>0)$ consists of $(k+1)^n$ closed, pairwise disjoint intervals $I_{n,r}$ of length $\pi/(k^n u_n)$ whose mutual distance is at least $3\pi/(5k^n u_{n-1})$. (We always assume that $\operatorname{Max} I_{n,r}<\operatorname{Min} I_{n,r+1}$ for all n and $r\in\{1, ..., (k+1)^n-1\}$.)

This fact is easily proved by induction, remarking that since $k\equiv 1 \pmod 4$ and $u_n\equiv 1 \pmod 4$, denoting by $[a, b]$ any interval $I_{n,r}$, we have

$$\cos(k^{n+1}u_n a)=\cos(k^{n+1}u_n b)=0,$$

$$\sin(k^{n+1}u_n a)= -1, \qquad \sin(k^{n+1}u_n b)=1.$$

Then $\cos(k^{n+1}u_n\theta)$ vanishes at $\theta=a+i(b-a)/k$ $(i=0, ..., k)$ in $[a, b]$ and takes the value 1 exactly $(k+1)/2$ times. In other words, $[a, b]$ consists of exactly k half-periods of $\theta\to\cos(k^{n+1}u_n\theta)$ and this function vanishes at a and b and is positive on the first, the third, the fifth, ..., the last half-period. Then $C_{n+1}\cap[a, b]$ consists of $(k+1)$ intervals of length $\pi/(k^{n+1}u_{n+1})$.

The distance between any two intervals is $\pi/(k^{n+1}u_n)-2\pi/(k^{n+1}u_{n+1})$. Let us note that

$$\pi/(k^{n+1}u_n)-2\pi/(k^{n+1}u_{n+1})\geqslant 3\pi/(5k^{n+1}u_n) \tag{2.3}$$

since $u_{n+1}\geqslant 5u_n$.

We put $C=C(k, \varphi)=\bigcap_{n=0}^{\infty} C_n$. The set $C(k, \varphi)$ is a symmetric Cantor set containing $\pi/2$.

3. Concerning the Regularity of $C(k, \varphi)$

In this section we show that if the function φ does not increase too fast then the set $C(k, \varphi)$ can be chosen to be regular. For such a result to hold, some restrictions on φ are necessary, since we have the following

Remark. Suppose that C is a compact subset of $I=[-1, 1]$ with a positive logarithmic capacity $\operatorname{cap}(C)$. Let G_C denote Green's function of $\mathbb{C}\setminus C$ with pole at ∞. Then $A:=\operatorname{Sup}_I \exp(G_C)$ is finite and by both the classical Markov inequality and the Bernstein–Walsh inequality, for each polynomial P of degree at most n, we have

$$\|P'\|_C\leqslant \|P'\|_I\leqslant n^2\,\|P\|_I\leqslant n^2A^n\,\|P\|_C.$$

Thus,

$$\|P'\|_C \leqslant \varphi(n)\,\|P\|_E$$

for any function φ satisfying $\varphi(t) \geqslant t^2 A^t$.

We prove however the following

PROPOSITION. *Suppose that $\varphi(t) \leqslant Mt^p$ with $M > 0$ and $p > 1$. Then one can choose k so that $C(k, \varphi)$ is regular with respect to Green's function of $\mathbb{C} \setminus C$ with pole at ∞.*

Proof. We adapt a reasoning of Tsuji [8, Theorem III.63]. Fix $n_0 \in \mathbb{N}$ and $r \in \{1, ..., (k+1)^{n_0}\}$ and put $J_0 = I_{n_0, r}$. Then

$$C \cap J_0 = \bigcap_{n=n_0+1}^{\infty} C_n \cap J_0 = \bigcap_{n=1}^{\infty} \bigcap_{r=1}^{(k+1)^n} J_{n,r},$$

where we put $J_{n,r} = I_{n_0+n,r} \cap J_0$, if $I_{n_0+n,r} \subset J_0$.

We note that the length of J_0, $|J_0| = \pi/(k^{n_0} u_{n_0})$ and

$$|J_{n,r}| = L_{n_0+n} = \pi/(k^{n_0+n} u_{n_0+n}) \tag{3.1}$$

for $n = 1, 2, ...$ and $1 \leqslant r \leqslant (k+1)^n$ and by (2.3)

$$d_n^0 := \operatorname{dist}(J_{n,r}, J_{n,s}) = d_{n_0+n} \geqslant 3\pi/(5k^{n_0+n} u_{n_0+n-1}), \tag{3.2}$$

if $1 \leqslant r < s \leqslant (k+1)^n$.

We put

$$E_n = \bigcup_{r=1}^{(k+1)^n} J_{n,r}$$

and take N points $X_i^{n,r}$ $(i = 1, ..., N)$ on each $J_{n,r}$ such that

$$\left(\prod_{1 \leqslant i < j \leqslant N} |X_i^{n,r} - X_j^{n,r}| \right)^{1/\binom{N}{2}} \searrow d(J_{n,r}) \qquad \text{for} \quad N \to \infty, \tag{3.3}$$

$d(J_{n,r})$ denoting the transfinite diameter of $J_{n,r}$ (see, e.g., [8, p. 71]). Since there are $(k+1)^n N$ points $X_i^{n,r}$ on E_n, we have by the definition of the transfinite diameter

$$[d_{N(k+1)^n}(E_n)]^{\binom{N(k+1)^n}{2}}$$

$$:= \operatorname{Max}\left\{ \left(\prod_{1 \leqslant i < j \leqslant N(k+1)^n} |y_i - y_j| \right); y_i, y_j \in E_n \right\}$$

$$\geqslant \prod_{r,s=1}^{(k+1)^n} \prod_{i,j=1}^{N} |X_i^{n,r} - X_j^{n,s}| =: P, \tag{3.4}$$

where we assume that $r \leqslant s$ and $i < j$ if $r = s$.

Then P consists of $(n+1)$ factors, $P=P_n P_{n-1} \cdots P_0$, where P_n is formed with pairs of points which lie in the same $J_{n,r}$ and P_{n-1} is formed with pairs of points which lie in the same $J_{n-1,r}$ and belong to $J_{n,s}$ and $J_{n,s'}$ $(s<s')$, respectively, and P_{n-2} is formed with pairs of points which lie in the same $J_{n-2,r}$ and belong to $J_{n-1,s}$ and $J_{n-1,s'}$ $(s<s')$, respectively, and finally P_0 is formed of pairs of points which lie in J_0 and belong to $J_{1,s}$ and $J_{1,s'}$ $(s<s')$, respectively.

By (3.3),

$$P_n^{1/\binom{N}{2}} \searrow d(J_{n,1}) \cdots d(J_{n,(k+1)^n}) \qquad \text{as} \quad N \to \infty. \tag{3.5}$$

Since the transfinite diameter of any interval I is $|I|/4$, we have by (3.1) $d(J_{n,r}) = \pi/(4k^{n_0+n} u_{n_0+n})$ so that

$$P_n \geqslant (\pi/(4k^{n_0+n} u_{n_0+n}))^{N^2(k+1)^n k/2} \qquad \text{if} \quad N > k+1. \tag{3.6}$$

By (3.2), if $X_i^{n,s}, X_j^{n,s'} \in J_{n-1,r}$ for some r and $s<s'$, then $|X_i^{n,s} - X_j^{n,s'}| \geqslant 3\pi/(5k^{n_0+n} u_{n_0+n-1})$ and the number of such pairs is $(k+1)^{n-1} \binom{k+1}{2} N^2$ so that

$$P_{n-1} \geqslant (3\pi/(5k^{n_0+n} u_{n_0+n-1}))^{N^2(k+1)^n k/2}. \tag{3.7}$$

Similarly, if $|X_i^{n,s} - X_j^{n,s'}|$ is a factor of P_{n-2} then

$$|X_i^{n,s} - X_j^{n,s'}| \geqslant d_{n-1}^0 \geqslant 3\pi/(5k^{n_0+n-1} u_{n_0+n-2})$$

and the number of factors of P_{n-2} is $(k+1)^{n-2} \binom{k+1}{2} [(k+1)N]^2$, so that

$$P_{n-2} \geqslant (3\pi/(5k^{n_0+n-1} u_{n_0+n-2}))^{N^2(k+1)^{n+1} k/2}, \tag{3.8}$$

and finally,

$$P_0 \geqslant (3\pi/(5k^{n_0+1} u_{n_0}))^{N^2(k+1)^{2n-1} k/2}. \tag{3.9}$$

By the assumption of the proposition, it follows from the definition of u_n that, for each $n \in \mathbb{N}$,

$$u_{n+1} \leqslant Mn^2 k^{p(n+1)} u_n^p \leqslant Mk^{2pn} u_n^p.$$

Therefore, one easily checks that

$$u_{n_0+t} \leqslant A^{p^t} k^{mn_0 p^t} u_{n_0}^{p^t}, \tag{3.10}$$

where $A = M^{1/(p-1)}$ and $m = 2p^2/(p-1)^2$.

Hence by (3.6), if $N > k+1$, we get

$$P_n \geqslant [2k^{n_0+n}(Ak^{mn_0}u_{n_0})^{p^n}]^{-kN^2(k+1)^n\,2} \tag{3.11}$$

and by (3.7), (3.8), (3.9), and (3.10)

$$\begin{aligned} P' := P_{n-1}P_{n-2}\cdots P_0 &\geqslant [k^{n_0+n}(Ak^{mn_0}u_{n_0})^{p^{n-1}}]^{-kN^2(k+1)^n\,2} \\ &\times [k^{n_0+n-1}(Ak^{mn_0}u_{n_0})^{p^{n-2}}]^{-kN^2(k+1)^{n+1}\,2} \times \cdots \\ &\times [k^{n_0+1}(Ak^{mn_0}u_{n_0})^{p^0}]^{-kN^2(k+1)^{2n-1}\,2}. \end{aligned}$$

Now if we take $k+1>2p$, by an easy calculation we get

$$P' > [k^{n_0+1+1/k}(Ak^{mn_0}u_{n_0})^2]^{-N^2(k+1)^{2n}\,2}. \tag{3.12}$$

Consequently, since

$$\binom{N(k+1)^n}{2} \sim \frac{(k+1)^{2n}N^2}{2}, \qquad \text{if} \quad N\to\infty,$$

from (3.4) we get for $k+1>2p$,

$$\begin{aligned} d(E_n) = \operatorname*{Lim}_{N\to\infty} d_{N(k+1)^n}(E_n) &\geqslant \operatorname*{Lim}_{N\to\infty} (P_nP')^{2N^{-2}(k+1)^{-2n}} \\ &> [2k^{n_0+n}(Ak^{mn_0}u_{n_0})^{p^n}]^{-k\,(k+1)^n}\,[k^{n_0+2}(Ak^{mn_0}u_{n_0})^2]^{-1}. \end{aligned}$$

Since $d(E_n)\to d(C\cap J_0)$ if $n\to\infty$, it follows that

$$d(C\cap J_0) \geqslant 1/[k^{n_0+2}(Ak^{mn_0}u_{n_0})^2] \geqslant B^{n_0}\,|J_0|^2, \tag{3.13}$$

where B is a positive constant depending only on k and p.

Fix now a point $a\in C=C(k,\varphi)$. Then there exists a sequence $I_n := I_{n,r_n}$ $(n=1,2,...)$, where $r_n\in\{1,...,(k+1)^n\}$, such that $a\in I_n$ for all n. Put $\varepsilon_n=|I_n|$. Then $C\cap I_n$ is contained in $K_n := C\cap\{|z-a|\leqslant\varepsilon_n\}$. By (3.13) we get

$$\begin{aligned} \operatorname*{Lim\,sup}_{n\to\infty} \frac{\operatorname{Ln}(1/\varepsilon_n)}{\operatorname{Ln}[1/d(K_n)]} &\geqslant \operatorname*{Lim\,sup}_{n\to\infty} \frac{\operatorname{Ln}(1/\varepsilon_n)}{\operatorname{Ln}[1/d(C\cap I_n)]} \\ &\geqslant \operatorname*{Lim}_{n\to\infty} \frac{\operatorname{Ln}(\varepsilon_n)}{n\operatorname{Ln}B+2\operatorname{Ln}\varepsilon_n} = 1/2. \end{aligned}$$

Thus, by Wiener's criterion (see [8, p. 104, Corollary 2 to Theorem III.62]), a is a regular point of C.

4. Counter-examples Proving the Theorem

(1) *Bernstein Inequality*

Let $T = T_{k^{n+1}u_n}$ be the trigonometric polynomial of order $k^{n+1}u_n$ defined by $T(\theta) = \cos(k^{n+1}u_n\theta)$. We have

$$\|T\|_{C(k,\varphi)} \leqslant \|T\|_{C_{n+1}} \leqslant \sin(\pi u_n/u_{n+1}) \leqslant \pi u_n/u_{n+1}$$

and

$$\|T'\|_{C(k,\varphi)} = |T'(\pi/2)| = k^{n+1}u_n;$$

therefore

$$\frac{\|T'\|_{C(k,\varphi)}}{\varphi(k^{n+1}u_n)\,\|T\|_{C(k,\varphi)}} \geqslant \frac{k^{n+1}u_n}{\varphi(k^{n+1}u_n)(\pi u_n/u_{n+1})} = \frac{k^{n+1}u_{n+1}}{\pi\varphi(k^{n+1}u_n)} .$$
$$\geqslant n/\pi \qquad \text{(see (2.2))}.$$

The last estimate shows that the Bernstein inequality on $C(k, \varphi)$ with coefficient φ is not satisfied for the polynomials $T_{k^{n+1}u_n}$.

If $\varphi(n) \leqslant Mn^p$ with $p > 1$, we choose $k + 1 > p$ and then by the proposition of Section 3, $C(k, \varphi)$ is regular.

(2) *Markov Inequality*

The Cantor set $C(k, \varphi)$ is symmetric with respect to 0 and does not contain 0. Then if $C(k, \varphi)$ is regular, so is the set $C(k, \varphi) \cap R^+$. The change of variable $x = \cos\theta$ maps $[0, \pi]$ onto $[-1, 1]$.

Let $D(k, \varphi) = \cos(C(k, \varphi) \cap \mathbb{R}^+)$. The set $D(k, \varphi)$ is a Cantor type set containing 0. By [4, Theorem 3.5], if $C(k, \varphi)$ is regular, so is the set $D(k, \varphi)$.

Let $P(x) = P_{k^{n+1}u_n}(x) = T_{k^{n+1}u_n}(\text{Arccos}(x))$ be the Chebyshev polynomial of degree $k^{n+1}u_n$. We have

$$\|P\|_{D(k,\varphi)} = \|T\|_{C(k,\varphi)} \leqslant \pi u_n/u_{n+1}$$

since $|P'(0)| = |T'(\pi/2)| = k^{n+1}u_n$, $\|P'\|_{C(k,\varphi)} \geqslant k^{n+1}u_n$ and by the previous argument,

$$\frac{\|P'\|_{D(k,\varphi)}}{\varphi(k^{n+1}u_n)\,\|P\|_{D(k,\varphi)}} \geqslant \frac{k^{n+1}u_{n+1}}{\pi\varphi(k^{n+1}u_n)} \geqslant n/\pi.$$

Then the Markov inequality on $D(k,\varphi)$ with coefficient φ is not satisfied for the polynomials $P_{k^{n+1}u_n}$.

ACKNOWLEDGMENTS

This work was written while the second author stayed at the University of Sciences and Technics of Lille–Flandres–Artois (France). He thanks the Laboratory of Complex Analysis of this university for the invitation and hospitality.

REFERENCES

1. P. GOETGHELUCK, Polynomial inequalities on general subsets of $\mathbb{R}^N$, *Colloq. Math.* 57 (1989), 127–136.
2. W. PAWŁUCKI AND W. PLEŚNIAK, Markov's inequality and C^∞ functions on sets with polynomial cusps, *Math. Ann.* **275** (1986), 467–480.
3. W. PAWŁUCKI AND W. PLEŚNIAK, Extension of C^∞ functions from sets with polynomial cusps, *Studia Math.* **88** (1988), 279–287.
4. W. PLEŚNIAK, Invariance of the L-regularity of the compacts sets in $\mathbb{C}^N$ under holomorphic mappings, *Trans. Amer. Math. Soc.* **246** (1978), 373–383.
5. W. PLEŚNIAK, Markov's inequality and the existence of an extension operator for C^∞ functions, *J. Approx. Theory* **61** (1990), 106–117.
6. W. PLEŚNIAK, A Cantor regular set which does not have Markov's property, *Ann. Polon. Math.* **51** (1990), 269–274.
7. Q. I. RAHMAN AND G. SCHMEISSER, "Les inégalités de Markoff et de Bernstein," Presses de l'Univ. de Montréal, Montréal, 1983.
8. M. TSUJI, "Potential Theory in Modern Function Theory," Maruzen, Tokyo, 1959.